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THE DIRECTED PACKING NUMBERS $DD(t, v, v)$, $t \geqq 4$

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A directed packing is a maximal collection of k-subsets, called blocks, of a set of cardinality v having the property that no ordered t-subset occurs in more than one block. A block contains an ordered t-set if its symbols appear, left to right, in the block. The cardinality of such a maximal collection is denoted by $DD(t, k, v)$. We consider the special case when $k=v$ and derive some results on the sizes of maximal collections.

1. Introduction

Directed packings are combinatorial structures which are used in the design of statistical experiments and large computer networks [3]. A directed packing is a maximal collection of blocks of size k whose elements are selected from a set of cardinality v with the restriction that no ordered t-subset occurs in more than one block. The block *abcd*, for example, is said to contain the four triples *abc*, *abd*, *acd* and *bcd*. The cardinality of the maximal collection is denoted by $DD(t, k, v)$ and the structure is called a (t, k, v) directed packing.

We examine a special case where k is equal to v. For (undirected) designs, coverings and packings this case is profoundly uninteresting as the structure will consist, in each case, of a single block containing all of the v symbols. However, when the blocks are ordered, non-trivial structures become possible. The $(3, v, v)$ directed packings have been examined in [2]. We obtain results which apply for larger t and give some $(4, v, v)$ and $(5, v, v)$ packings.

Some simple facts about directed packings of this kind $(k=v)$ can be observed. An upper bound, namely,

$$DD(t, v, v) \leqq t! \tag{1}$$

can be derived by counting the frequencies of t-sets. There are $v(v-1)...(v-t+1)$ possible ordered t-sets and $\binom{v}{t}$ of them can be packed into each block. Since no t-set may occur more than once the result follows.

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It is also clear that a directed packing containing two blocks, the first with symbols in arbitrary order and the second with the symbols in exactly reverse order can never contain a repeated t-set. Thus

(2) $$DD(t, v, v) \geqq 2$$

for all v. In fact, for any fixed t and v sufficiently large, this is the maximal directed packing as we show in section 2.

As v increases, the number of blocks in a directed packing cannot increase. Formally

(3) $$DD(t, v+i, v+i) \leqq DD(t, v, v)$$

for all non-negative i. This is because deleting i symbols from a $(t, v+i, v+i)$ packing gives a structure with v symbols, $DD(t, v+i, v+i)$ blocks and certainly no repeated t-set.

2. The Erdős—Szekeres Theorem

A very old result (1935, [1]) due to Erdős and Szekeres concerns sequences containing increasing or decreasing subsequences. Let $\gamma(i, j)$ be the minimum number of symbols such that writing them down in any order will result in a sequence containing either an increasing sequence of i symbols or a decreasing sequence of j symbols.

Theorem 1 [Erdős—Szekeres].

$$\gamma(i, j) = (i-1)(j-1)+1. \quad \blacksquare$$

This bound is exact so that it is always possible to write down $\gamma(i, j)-1$ symbols without either an increasing or decreasing sub-sequence of the appropriate length. We use this to establish

Theorem 2. *If* $v \geqq (t-1)^3+1$ *then* $DD(t, v, v)=2$.

Proof. Suppose $v \geqq \gamma(t, \gamma(t, t))=(t-1)^3+1$ and we attempt to construct a directed packing with more than two blocks. Without loss of generality, the first block may be $123\ldots v$. Any second block may not contain an increasing t-set and thus must contain a decreasing subsequence of length $\gamma(t, t)$. The symbols in this subsequence must contain either an increasing or decreasing subsequence of length t in any third block since there are $\gamma(t, t)$ of them and such a t-sequence is a repeat of one in block 1 or block 2. Thus there can be at most two blocks and, from (2), at least two blocks proving the result. $\blacksquare$

3. A better lower bound

Theorem 3. *If* $v \leqq (t-1)^3$ *then* $DD(t, v, v) \geqq 4$.

Proof. Consider the sets

$$\{(a, b, c)\}, \ \{(Ra, Rb, c)\}, \ \{(Ra, b, Rc)\}, \ \{(a, Rb, Rc)\},$$

where $a, b, c \in \{1, 2, \ldots, t-1\}$ and R is the transformation taking i to $t-i$

$(i = 1, 2, \ldots, t-1)$. If these four sets of $(t-1)^3$ triples are written with the triples (a, b, c) appearing in lexicographic order, and the other sets of triples in corresponding order, then they form blocks containing symbols from a set of cardinality $(t-1)^3$ and not containing any repeated t-tuple. ∎

An example will illustrate the procedure. Suppose that $t=4$. Then the resulting four blocks are shown below; a numering of the triple from 1 to 27 is also shown.

Block 1		Block 2		Block 3		Block 4	
111	1	331	25	313	21	133	9
112	2	332	26	312	20	132	8
113	3	333	27	311	19	131	7
121	4	321	22	323	24	123	6
122	5	322	23	322	23	122	5
123	6	323	24	321	22	121	4
131	7	311	19	333	27	113	3
132	8	312	20	332	26	112	2
133	9	313	21	231	25	111	1
211	10	231	16	213	12	233	18
212	11	232	17	212	11	232	17
213	12	233	18	211	10	231	16
221	13	221	13	223	15	223	15
222	14	222	14	222	14	222	14
223	15	223	15	221	13	221	13
231	16	211	10	233	18	213	12
232	17	212	11	232	17	212	11
233	18	213	12	231	16	211	10
311	19	131	7	113	3	333	27
312	20	132	8	112	2	332	26
313	21	133	9	111	1	331	25
321	22	121	4	123	6	323	24
322	23	122	5	122	5	322	23
323	24	123	6	121	4	321	22
331	25	111	1	133	9	313	21
332	26	112	2	132	8	312	20
333	27	113	3	131	7	311	19

This shows that the Erdős—Szekeres bound is exact and gives the smallest value of v such that only two sets can be formed. It can be generalized to give:

Theorem 4. *If $v=p^q$ then there are n blocks of size v which contain no repeated $(p^{q-d}+1)$-set, where n is the maximum cardinality of a set C of elements of $\{1, R\}^q$ with minimum Hamming distance d.*

Proof. Let B be the block of v elements formed by taking all q-tuples of the form $(a_1, a_2, \ldots, a_q)$, where $a_i \in \{1, 2, \ldots, p\}$, in lexicographic order. The elements of C are q-tuples of transformations which are at a Hamming distance at least d from each other; thus, for each $R^* \in C$, a new block $R^*(B)$ can be formed by applying $R^* \in \{1, R\}^q$ to each q-tuple $(a_1, a_2, \ldots, a_q)$ in B.

To show that two blocks contain no common t-tuple $(t=p^{q-d}+1)$ let B be the block consisting of all q-tuples taken in lexicographic order, and let R^* and $S^* \in C$; we show that $S^*(B)$ and $R^*(B)$ have no t-tuple in common. Consider any t-subset of

$\{1, 2, \ldots, p\}^q$. Then as $t > p^{q-d}$, there must be two members of the t-subset which agree in those positions where S^* and R^* agree. Importantly, the first position where these two t-tuples disagree is one where R^* and S^* disagree. Hence they appear in the other order in S^* than in R^*. As this is true of any t-set of q-tuples, the two blocks have no common t-tuple. ▮

We illustrate this with an example. Suppose that $v=16$, $p=2$, $q=4$. Then the elements of $\{1, R\}^4$ with minimum Hamming distance 2 are $(1, 1, 1, 1)$, $(1, 1, R, R)$, $(1, R, 1, R)$, $(1, R, R, 1)$, $(R, 1, 1, R)$, $(R, 1, R, 1)$, $(R, R, 1, 1)$, (R, R, R, R). Thus if we form the block {0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111} and apply the eight transformations above to it, we obtain eight blocks which are, written horizontally in decimal,

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

It is easy to see that this collection does not contain a repeated 5-set.

This theorem gives a number of lower bounds on the number of blocks possible. The table which follows gives some lower bounds based on Theorems 3 and 4, and on the specific examples of packings given in the next section.

Table 1

Lower Bounds on $DD(t, v, v)$

t \ v	4	5	6	7	8	9	11	16	25	27	32	64	81	128
3	6				4	2→								
4		24	15	12		8	6			4	2→			
5			120	63	48	27		8				4	2→	
9											16			8
10													8	
17												32		
28													16	
33														64

Notice that the table entries decrease across each row, from (3), and increase down each column since a collection of blocks containing no repeated t-triple certainly contains no repeated $(t+1)$-triple. Thus lower bounds on the number of blocks in other packings are implied.

4. The Packing Numbers $DD(4, v, v)$ and $DD(5, v, v)$

From the results of the previous sections the following facts follow:

1. $DD(4, v, v) \leqq 24$,
2. $DD(4, v, v) \geqq 4 \quad \text{for} \quad v \leqq 27$,
3. $DD(4, v, v) = 2 \quad \text{for} \quad v > 27$.

We show first that $DD(4, 5, 5) = 24$. If this is the case then every ordered 4-set must occur exactly once. Let i be the number of times a symbol appears in the first position, j the number of times it appears in the second position and k, l, m the number of times it occurs in the third, fourth and fifth positions respectively. Then i, j, k, l, m must satisfy the following set of equations.

$$\begin{aligned} 4i+j &= 24 \\ 3j+2k &= 24 \\ 2k+3l &= 24 \\ l+4m &= 24 \end{aligned}$$

This set of equations has three solutions:

Type I: $i = 4$, $j = 8$, $k = 0$, $l = 8$, $m = 4$.

Type II: $i = 5$, $j = 4$, $k = 6$, $l = 4$, $m = 5$.

Type III: $i = 6$, $j = 0$, $k = 12$, $l = 0$, $m = 6$.

Since every position in every block must be filled, if there are A points of type I, B points of type II and C points of type III then A, B and C must satisfy the following equations:

$$\begin{aligned} 4A+5B+6C &= 24 \\ 8A+4B &= 24 \\ 6B+12C &= 24 \end{aligned}$$

giving the three solutions

(a) $A = 1$, $B = 4$, $C = 0$

(b) $A = 2$, $B = 2$, $C = 1$

(c) $A = 3$, $B = 0$, $C = 2$.

Suppose that in a solution of type (c), a and b are the symbols of type III. All blocks have either a or b in the centre position, and thus there are six blocks of the form $axbxx$. This gives twelve quadruples of the form $axbx$, but there are only six different such quadruples. Hence there is no solution of type (c).

In considering solutions of type (a) and (b) we take the 24 permutations of $\{a, b, c, d\}$ (calling these *4-blocks*) and insert e in each one (forming a *block*), avoiding any repeated quadruple. We find four essentially distinct designs.

Firstly we look at solutions of type (b). Let a and b be the symbols of type *I*, c and d of type *II* and e of type III. Now, for any four 4-blocks of the pattern

acbd acdb cabd cadb

we can insert e (in positions 1, 3 or 5) in only one way. Since a and b do not appear in the third position of a block, *caebd* is a block and *acbde* is not. Also, *acedb* is not a block (for otherwise we have a repeated quadruple, *aebd*) and so *eacbd* is a block. Similarly *cadbe* is a block. Again, avoiding repeated quadruples requires *acedb* to be a block, and we have

eacbd acedb caebd cadbe.

However, for the four 4-blocks

abcd abdc bacd badc

we can insert e in two ways, to get either

abecd abdce bacde baedc, or

abcde abedc baecd badce.

We have a similar choice for the four 4-blocks

cdab cdba dcab dcba.

The four designs resulting are the two shown below and the two got from these by interchanging c and d.

(1)

e	*a*	*c*	*b*	*d*	*d*	*b*	*c*	*a*	*e*
a	*c*	*e*	*d*	*b*	*b*	*d*	*e*	*c*	*a*
c	*a*	*e*	*b*	*d*	*d*	*b*	*e*	*a*	*c*
c	*a*	*d*	*b*	*e*	*e*	*b*	*d*	*a*	*c*
e	*a*	*d*	*b*	*c*	*c*	*b*	*d*	*a*	*e*
a	*d*	*e*	*c*	*b*	*b*	*c*	*e*	*d*	*a*
d	*a*	*e*	*b*	*c*	*c*	*b*	*e*	*a*	*d*
d	*a*	*c*	*b*	*e*	*e*	*b*	*c*	*a*	*d*
a	*b*	*c*	*d*	*e*	*e*	*d*	*c*	*b*	*a*
a	*b*	*e*	*d*	*c*	*c*	*d*	*e*	*b*	*a*
b	*a*	*e*	*c*	*d*	*d*	*c*	*e*	*a*	*b*
b	*a*	*d*	*c*	*e*	*e*	*c*	*d*	*a*	*b*

(2)

e	*a*	*c*	*b*	*d*	*d*	*b*	*c*	*a*	*e*
a	*c*	*e*	*d*	*b*	*b*	*d*	*e*	*c*	*a*
c	*a*	*e*	*b*	*d*	*d*	*b*	*e*	*a*	*c*
c	*a*	*d*	*b*	*e*	*e*	*b*	*d*	*a*	*c*
e	*a*	*d*	*b*	*c*	*c*	*b*	*d*	*a*	*e*
a	*d*	*e*	*c*	*b*	*b*	*c*	*e*	*d*	*a*
d	*a*	*e*	*b*	*c*	*c*	*b*	*e*	*a*	*d*
d	*a*	*c*	*b*	*e*	*e*	*b*	*c*	*a*	*d*
a	*b*	*c*	*d*	*e*	*d*	*c*	*e*	*b*	*a*
a	*b*	*e*	*d*	*c*	*e*	*c*	*d*	*b*	*a*
b	*a*	*e*	*c*	*d*	*e*	*d*	*c*	*a*	*b*
b	*a*	*d*	*c*	*e*	*c*	*d*	*e*	*a*	*b*

In checking that these have no repeated quadruple, the reader may note that a block with e in position i cannot have a quadruple in common with a block with e in position j if $|j-i|>1$.

We now consider solutions of type (a), letting e be the element of type I. Of the six 4-blocks of the form $axxx$, let $n(a)$ have e inserted in position 2. As there may be no repeated quadruple, $n(a) \leqq 2$ (e.g. $abcd$ and $adcb$, becoming $aebcd$ and $aedcb$) and these two 4-blocks must have the same element in position 3. But as $n(a)+n(b)+n(c)+n(d)=8$, $n(a)=2$ etc. Thus the eight 4-blocks to which e is inserted in position 2 consist of two with each of the pairs $(a, \theta(a))$, $(b, \theta(b))$, $(c, \theta(c))$ and $(d, \theta(d))$, say, in positions (1, 3). We show that θ is a bijection. Suppose, on the contrary, that $\theta(a)=\theta(b)=d$. Since d appears in the fourth position of exactly four blocks, $\theta(c) \neq d$, and also there are no blocks of the form $exxdx$. Thus the one block $cexxd$ and k blocks of the form $exxxd$ give $3k+1$ quadruples of the form $exxd$ and so we cannot get the six quadruples once each. Hence this case is impossible.

Similarly we look at the pairs of elements $(\varphi(x), x)$ in positions (2,4) of those 4-blocks into which e is inserted in position 4. Given θ, the choice of φ is limited by the fact that no $(x, \varphi(y), \theta(x), y)$ may be a 4-block. We have three possible patterns.

(a)				(b)				(c)			
x	θx	φy	y	x	θx	φy	y	x	θx	φy	y
a	b	a	c	a	d	a	c	a	c	d	a
b	a	c	a	b	a	c	a	c	a	a	b
c	d	b	d	c	b	b	d	b	d	b	c
d	c	d	b	d	c	d	b	d	b	c	d

In each case, for each of the eight 4-blocks remaining, there is only one choice as to whether e is inserted in position 1 or 5, and a design with no repeated quadruples results. As each of these three patterns can be produced in 6 ways, there are eighteen such designs. The designs from patterns (b) and (c) are obtained from each other by reversing blocks. The designs from patterns (a) and (b) are given below.

(3)										(4)									
a	e	c	b	d	b	a	d	e	c	a	e	b	d	c	b	a	d	e	c
a	e	d	b	c	d	a	b	e	c	a	e	c	d	b	d	a	b	e	c
b	e	c	a	d	b	c	d	e	a	b	e	c	a	d	b	c	d	e	a
b	e	d	a	c	d	c	b	e	a	b	e	d	a	c	d	c	b	e	a
c	e	a	d	b	a	b	c	e	d	c	e	a	b	d	a	b	c	e	d
c	e	b	d	a	c	b	a	e	d	c	e	d	b	a	c	b	a	e	d
d	e	a	c	b	a	d	c	e	b	d	e	a	c	b	a	d	c	e	b
d	e	b	c	a	c	d	a	e	b	d	e	b	c	a	c	d	a	e	b
e	a	b	d	c	a	c	d	b	e	e	a	d	b	c	a	c	b	d	e
e	b	a	c	d	c	a	b	d	e	e	b	a	c	d	b	d	c	a	e
e	c	d	b	a	b	d	c	a	e	e	c	b	d	a	c	a	d	b	e
e	d	c	a	b	d	b	a	c	e	e	d	c	a	b	d	b	a	c	e

Therefore, up to interchanging symbols and reversing all blocks, there are four distinct designs, and $DD(5, 6, 6)=24$.

The possibility of establishing many more results computationally seems remote. Essentially, determining $DD(t, v, v)$ involves searching a tree of depth $t!$ with order $v!$ possibilities (all permutations on v symbols) at each node and for any but very small t and v this is impractical.

Table 2

$v=6$

1 2 3 4 5 6	3 6 1 5 2 4
1 2 3 6 5 4	3 6 2 4 1 5
1 2 5 4 6 3	3 6 2 5 1 4
1 3 2 4 6 5	3 6 5 4 2 1
1 3 2 5 6 4	4 1 2 6 5 3
1 3 6 4 5 2	4 1 3 5 6 2
1 4 2 5 3 6	4 2 1 3 6 5
1 4 2 6 3 5	4 2 1 5 6 3
1 4 3 5 2 6	4 2 3 5 6 1
1 4 3 6 2 5	4 3 1 2 5 6
1 4 5 6 2 3	4 3 1 6 5 2
1 4 6 5 3 2	4 3 2 6 5 1
1 5 2 4 3 6	4 5 1 2 3 6
1 5 2 6 3 4	4 5 1 6 3 2
1 5 3 4 2 6	4 5 2 6 3 1
1 5 3 6 2 4	4 5 3 2 1 6
1 5 6 4 3 2	4 5 3 6 1 2
1 6 2 4 3 5	4 6 1 3 2 5
1 6 2 5 3 4	4 6 1 5 2 3
1 6 3 4 2 5	4 6 2 3 1 5
1 6 3 5 2 4	4 6 2 5 1 3
1 6 5 4 2 3	4 6 3 5 2 1
2 1 3 5 4 6	5 1 2 6 4 3
2 1 6 4 5 3	5 1 3 4 6 2
2 3 1 4 6 5	5 2 1 3 6 4
2 3 1 5 6 4	5 2 1 4 6 3
2 3 6 4 5 1	5 2 3 4 6 1
2 4 1 5 3 6	5 3 1 2 4 6
2 4 1 6 3 5	5 3 1 6 4 2
2 4 3 5 1 6	5 3 2 6 4 1
2 4 3 6 1 5	5 4 1 3 2 6
2 4 5 6 1 3	5 4 1 6 2 3
2 4 6 5 3 1	5 4 2 3 1 6
2 5 1 4 3 6	5 4 2 6 1 3
2 5 1 6 3 4	5 4 3 6 2 1
2 5 3 4 1 6	5 4 6 3 1 2
2 5 3 6 1 4	5 6 1 3 2 4
2 5 6 4 3 1	5 6 1 4 2 3
2 6 1 4 3 5	5 6 2 3 1 4
2 6 1 5 3 4	5 6 2 4 1 3
2 6 3 4 1 5	5 6 3 4 2 1
2 6 3 5 1 4	6 1 2 4 5 3
2 6 5 4 1 3	6 1 3 5 4 2
3 1 2 6 4 5	6 2 1 3 4 5
3 1 5 4 6 2	6 2 1 5 4 3
3 2 1 4 5 6	6 2 3 5 4 1
3 2 1 6 5 4	6 3 1 2 5 4
3 2 5 4 6 1	6 3 1 4 5 2
3 4 1 5 2 6	6 3 2 4 5 1
3 4 1 6 2 5	6 4 1 2 3 5
3 4 2 5 1 6	6 4 1 5 3 2
3 4 2 6 1 5	6 4 2 5 3 1
3 4 5 6 2 1	6 4 3 2 1 5
3 4 6 5 1 2	6 4 3 5 1 2
3 5 1 4 2 6	6 4 5 2 1 3
3 5 1 6 2 4	6 5 1 2 3 4
3 5 2 4 1 6	6 5 1 4 3 2
3 5 2 6 1 4	6 5 2 4 3 1
3 5 6 4 1 2	6 5 3 2 1 4
3 6 1 4 2 5	6 5 3 4 1 2

$v=7$

1 2 3 4 6 7 5
2 3 4 5 7 1 6
3 4 5 6 1 2 7
4 5 6 7 2 3 1
5 6 7 1 3 4 2
6 7 1 2 4 5 3
7 1 2 3 5 6 4
2 4 6 1 5 7 3
3 5 7 2 6 1 4
4 6 1 3 7 2 5
5 7 2 4 1 3 6
6 1 3 5 2 4 7
7 2 4 6 3 5 1
1 3 5 7 4 6 2
4 1 5 2 3 7 6
5 2 6 3 4 1 7
6 3 7 4 5 2 1
7 4 1 5 6 3 2
1 5 2 6 7 4 3
2 6 3 7 1 5 4
3 7 4 1 2 6 5

1 2 5 4 7 3 6
2 3 6 5 1 4 7
3 4 7 6 2 5 1
4 5 1 7 3 6 2
5 6 2 1 4 7 3
6 7 3 2 5 1 4
7 1 4 3 6 2 5
2 4 3 1 7 6 5
3 5 4 2 1 7 6
4 6 5 3 2 1 7
5 7 6 4 3 2 1
6 1 7 5 4 3 2
7 2 1 6 5 4 3
1 3 2 7 6 5 4
4 1 6 2 7 5 3
5 2 7 3 1 6 4
6 3 1 4 2 7 5
7 4 2 5 3 1 6
1 5 3 6 4 2 7
2 6 4 7 5 3 1
3 7 5 1 6 4 2

1 4 3 7 5 2 6
2 5 4 1 6 3 7
3 6 5 2 7 4 1
4 7 6 3 1 5 2
5 1 7 4 2 6 3
6 2 1 5 3 7 4
7 3 2 6 4 1 5
2 1 6 7 3 4 5
3 2 7 1 4 5 6
4 3 1 2 5 6 7
5 4 2 3 6 7 1
6 5 3 4 7 1 2
7 6 4 5 1 2 3
1 7 5 6 2 3 4
4 2 5 7 6 1 3
5 3 6 1 7 2 4
6 4 7 2 1 3 5
7 5 1 3 2 4 6
1 6 2 4 3 5 7
2 7 3 5 4 6 1
3 1 4 6 5 7 2

$v=8$

8 6 7 2 4 3 5 1
4 3 5 1 8 6 7 2
7 2 8 6 5 1 4 3
2 7 6 8 1 5 3 4
6 8 2 7 3 4 1 5
5 1 4 3 7 2 8 6
3 4 1 5 6 8 2 7
1 5 3 4 2 7 6 8

4 7 5 8 2 3 6 1
8 5 7 4 1 6 3 2
5 8 4 7 6 1 2 3
1 6 3 2 8 5 7 4
3 2 1 6 7 4 8 5
7 4 8 5 3 2 1 6
6 1 2 3 5 8 4 7
2 3 6 1 4 7 5 8

2 5 6 4 8 3 7 1
1 7 3 8 4 6 5 2
6 4 2 5 7 1 8 3
8 3 7 1 2 5 6 4
7 1 8 3 6 4 2 5
3 8 1 7 5 2 4 6
5 2 4 6 3 8 1 7
4 6 5 2 1 7 3 8

8 2 5 1 3 4 6 7
4 1 7 2 6 8 3 5
7 6 4 3 1 5 2 8
2 8 3 4 5 1 7 6
6 7 1 5 4 3 8 2
5 3 8 6 2 7 1 4
3 5 2 7 8 6 4 1
1 4 6 8 7 2 5 3

4 8 6 1 3 2 7 5
8 4 3 2 6 1 5 7
5 7 2 3 1 6 8 4
1 2 7 4 5 8 6 3
3 6 8 5 4 7 2 1
7 5 1 6 2 3 4 8
6 3 4 7 8 5 1 2
2 1 5 8 7 4 3 6

2 4 7 1 3 8 5 6
1 8 5 2 6 4 7 3
6 5 8 3 1 7 4 2
8 1 6 4 5 2 3 7
7 3 2 5 4 6 1 8
3 7 4 6 2 5 8 1
5 6 1 7 8 3 2 4
4 2 3 8 7 1 6 5

$v=9$

7 6 3 4 8 1 9 2 5
9 8 7 2 5 4 3 1 6
4 9 1 5 3 8 2 6 7
3 5 9 1 6 2 7 4 8
8 4 6 3 2 9 5 7 1
6 1 5 9 4 7 8 3 2
1 7 2 8 9 6 4 5 3
5 2 8 7 1 3 6 9 4
2 3 4 6 7 5 1 8 9

3 6 9 2 8 4 7 1 5
7 8 3 1 5 2 9 4 6
1 9 2 6 3 5 4 8 7
9 5 7 4 6 1 3 2 8
6 4 5 7 2 3 8 9 1
5 1 8 3 4 9 6 7 2
2 7 4 5 9 8 1 6 3
8 2 6 9 1 7 5 3 4
4 3 1 8 7 6 2 5 9

9 6 7 1 8 2 3 4 5
3 8 9 4 5 1 7 2 6
2 9 4 8 3 6 1 5 7
7 5 3 2 6 4 9 1 8
5 4 8 9 2 7 6 3 1
8 1 6 7 4 3 5 9 2
4 7 1 6 9 5 2 8 3
6 2 5 3 1 9 8 7 4
1 3 2 5 7 8 4 6 9

It is known that $DD(4, 6, 6) \geqq 15$, $DD(4, 7, 7) \geqq 12$, $DD(4, 9, 9) \geqq 8$ and $DD(4, 11, 11) \geqq 6$ as the following packings show.

1	6	2	5	4	3
2	6	3	1	5	4
3	6	4	2	1	5
4	6	5	3	2	1
5	6	1	4	3	2

1	6	3	4	5	2
2	6	4	5	1	3
3	6	5	1	2	4
4	6	1	2	3	5
5	6	2	3	4	1

1	5	3	2	6	4
2	1	4	3	6	5
3	2	5	4	6	1
4	3	1	5	6	2
5	4	2	1	6	3

1	7	2	3	5	6	4
2	7	3	4	6	1	5
3	7	4	5	1	2	6
4	7	5	6	2	3	1
5	7	6	1	3	4	2
6	7	1	2	4	5	3

1	4	3	2	6	7	5
2	5	4	3	1	7	6
3	6	5	4	2	7	1
4	1	6	5	3	7	2
5	2	1	6	4	7	3
6	3	2	1	5	7	4

1	2	4	8	9	6	5	3	7
2	3	5	1	9	7	6	4	8
3	4	6	2	9	8	7	5	1
4	5	7	3	9	1	8	6	2
5	6	8	4	9	2	1	7	3
6	7	1	5	9	3	2	8	4
7	8	2	6	9	4	3	1	5
8	1	3	7	9	5	4	2	6

2	1	3	9	4	5	10	6	7	11	8
1	11	2	7	10	8	5	6	9	4	3
6	5	7	9	8	1	10	2	3	11	4
5	11	6	3	10	4	1	2	9	8	7
4	10	3	8	11	9	7	1	2	6	5
8	10	7	4	9	11	3	5	6	2	1

These packings were discovered by computer. A programme was written to search for packings in which there are one or more "initial" blocks, and the remaining blocks are obtained by adding 1 (modulo v) $v-1$ times. This having been done for $v=5$, 6 and 8, it was then easy to extend the packings obtained to get the first three above. The last one was found by starting with a packing of six blocks with $v=8$, and repeatedly using a program which finds all ways to extend a given packing by inserting the new element into each block.

From the previous section we have:

1. $DD(5, v, v) \leqq 120$
2. $DD(5, v, v) \geqq 4$ for $v \leqq 64$
3. $DD(5, v, v) = 2$ for $v > 64$.

Computer searches have shown that $DD(5, 6, 6) = 120$, $DD(5, 7, 7) \geqq 63$, $DD(5, 8, 8) \geqq 48$ and $DD(5, 9, 9) \geqq 27$. Note also that $DD(5, 16, 16) \geqq 8$, from Theorem 4.

The configuration with $v = 6$ was found using a computer program which took the 120 permutations of $\{1, 2, 3, 4, 5\}$ and found ways to insert 6 somewhere in each of them to obtain a (5, 6, 6) directed packing. The method of this program will be reported elsewhere [4].

The packing with $v = 7$ is found by taking the "initial" blocks 1234675, 1254736 and 1437526, multiplying each by 1, 2 and 4 (mod 7), and adding 1 (mod 7) 6 times.

For the packing with $v = 8$, the symbols represent elements of $GF(8)$, where the symbol i represents $x^i (i = 1, \ldots, 7)$, 8 represents the zero element, and $x + x^2 = x^4$. We take the "initial" blocks 15342768 and 14687253, take the square and fourth power of each element (these are automorphisms of $GF(8)$), and add each element, $1, \ldots, 8$, in turn. (Thus the 'initial' blocks appear in the 8th and 32nd positions.)

For the case $v = 9$, we look at $GF(9)$, although we only use its properties as a 3-dimensional vector space over $GF(3)$. We let i represent x^i, 9 represent the zero element, and let $x + x^3 = x^4$. To the "initial" block 132578469, we apply each power of the linear transformation (124)(3)(568)(7)(9) (as it appears written as a permutation in cycle form), and add each element 1 to 9 in turn. (Thus the "initial" block appears last).

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